Problem #1

We apply separation of variables

with $u(t,x) = w(t)\, v(x)$

$$\frac{w''(t)}{w(t)} = -\frac{v^{(4)}(x)}{v(x)} = \lambda$$

where $\lambda$ is a constant.

Since we are seeking the oscillatory solutions we look for solutions

of the form

$$u(x,t) = (A \sin(\omega t) + B \cos(\omega t))\, v(x)$$

$$\Rightarrow \lambda = \omega^2$$

The equation

$$v^{(4)} + \lambda v = 0$$

has a general solution

$$v(x) = C \cos(\sqrt{\omega}\, x) + D \sin(\sqrt{\omega}\, x) +$$

$$E \cosh(\sqrt{\omega}\, x) + F \cosh(\sqrt{\omega}\, x)$$

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implying that

$$v''(x) = -\omega C \cos(\sqrt{\omega}\, x) - \omega D \sin(\sqrt{\omega}\, x) + \omega E \cosh(\sqrt{\omega}\, x) + \omega F \sinh(\sqrt{\omega}\, x)$$

Applying the boundary conditions we obtain.

$$\left.\begin{aligned} 0 &= v(0) = C + E \\ 0 &= v''(0) = -\omega C + \omega E \end{aligned}\right\} \quad C = E = 0.$$

$$0 = v(1) = D \sin(\sqrt{\omega}) + F \sinh(\sqrt{\omega})$$

$$0 = v''(1) = -\omega D \sin(\sqrt{\omega}) + \omega F \sinh(\sqrt{\omega})$$

$$\Rightarrow F = 0.$$

$$\sin\sqrt{\omega} = 0 \iff \sqrt{\omega} = k\pi \quad k \in \mathbb{N}^+$$

$$v_k(x) = \sin(k\pi x)$$

where $\lambda_k = \omega_k^2 = k^4\pi^4$ as eigenvalues.

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(b) The initial condition $w'(0)=0$

$\Rightarrow A=0$ while from $u(0,x)=f(x)$

we obtain the Fourier coefficients

$$u(t,x)=\sum_{k=1}^{\infty} b_k \cos(k^2\pi^2 t)\sin(k\pi x)$$

and therefore

$$u(0,x)=\sum_{k=1}^{\infty} b_k \sin(k\pi x)=f(x)$$

which then gives

$$b_k = 2\int_0^1 f(x)\sin(k\pi x)\,dx$$

which means $u(t,x)$ is well defined.

## Problem 2

a) D'Alembert's Principle states that

$$u(x,t) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

if $u(x,0) = \phi(x)$
$u_t(x,t) = \psi(x)$

$$u(x,t) = \frac{1}{2}\left(e^{(x-ct)^2} + e^{(x+ct)^2}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\,ds$$

$$u(x,t) = \frac{1}{2}\left(e^{(x-ct)^2} + e^{(x+ct)^2}\right) + \frac{1}{2c}\left(-\cos(x+ct) + \cos(x-ct)\right)$$

b) The Fourier transform of $e^{-x^2}$ is $Ce^{-|\xi|^2}$ where $C$ is independent of $\xi$.

$\partial_{tt} \hat{u} = -c^2 |\xi|^2 \hat{u}$ so $e^{it\xi c}$ or $e^{-it\xi c}$ are solutions to this equation.

since differentiation in $x$ becomes multiplication by $\cdot i\xi$ under the Fourier transform.

$$\hat{u}(t,\xi) = C_1 e^{it\xi c} + C_2 e^{-it\xi c}$$

$$C_1 + C_2 = \widehat{e^{-x^2}} = Ce^{-|\xi|^2} \qquad i\xi c\, C_1 - i\xi c\, C_2 = 0$$

$$\Rightarrow C_1 = C_2$$

$$2C_1 = Ce^{-|\xi|^2}$$

$$C_1 = \frac{1}{2}\left(Ce^{-|\xi|^2}\right) = C_2$$

$$\hat{u}(t,\xi) = \frac{C}{2}\left(e^{-|\xi|^2} e^{it\xi c} + e^{-|\xi|^2 - it\xi c}\right)$$

Taking the inverse Fourier transfon we derive at.

$$u(t,x) = \frac{e^{-(x-ct)^2} + e^{(x+ct)^2}}{2}$$

which agrees with D'Alembert's Principle

Problem 3

a) Let $u \in C^2(\bar{\Omega})$ be a classical solution to the equation $\Delta u = 0$. Then $u$ reaches its maximum on the boundary of $\partial\Omega$ otherwise $u \equiv 0$ (or a constant). This maximum is strict in the sense that if $x_0 \in \Omega^\circ$ then $u(x_0) < u(x_*)$ if $x_* \in \partial\Omega$ is s.t. $u(x_*)$ is the maximum of $u$.

(b) Assume that $u(x)$ is not unique. Then $\exists\ u_1, u_2$ s.t.

$$\Delta u_1 = f \quad u_1\big|_{\partial\Omega} = h$$

and $$\Delta u_2 = f \quad u_2\big|_{\partial\Omega} = h.$$

with $u_1 \neq u_2$. It follows that

$$w = u_1 - u_2 \quad \text{solves} \quad \Delta w = 0 \quad w\big|_{\partial\Omega} = 0.$$

Since the maximum of $w$ must be reached on the boundary as $w$

is harmonic $w < 0$.

However integration by parts

$$\Delta w \, \overline{w} = 0$$

$$\Rightarrow \int_{\Omega} \Delta w \, \overline{w} = -\int_{\Omega} |\nabla w|^2 = 0$$

implies $w$ is constant everywhere.

This constant must necessarily be 0 by continuity of $w$.

This forces $u_1 = u_2$ and the solutions are unique.

Problem 4

a)

The characteristic curves are given by the following ODE

$$\frac{dx}{dt} = 1+x^2$$

the solution is $\arctan x = t+c$

$$x = \tan(t+c)$$

$c \in \mathbb{R}$ $\quad t+c \in (-\pi/2, \pi/2)$

b) The general solution is

$u(x,t) = F(t - \arctan(x))$

if $u(0,x) = f(x)$ this implies $F(\arctan x) = f(x)$.

c) $u(x,t) = f(\tan(t - \arctan x))$

$\Rightarrow F = f \circ \tan$

Solution isn't defined for $x < \tan(t - \pi/2)$

whenever $0 < t < \pi$ or $t \geq \pi$

as $t \to \pi$ waves go to $\infty$ and the solution goes to $0$.